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Euler Incognito

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Abstract

The nonlinear flow equations discussed recently by Bender and Feinberg are all reduced to the well-known Euler equation after a change of variables.

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Consider for $u(x, t)$ the class of nonlinear PDEs introduced and discussed by Bender and Feinberg [1]

$$u_t = (u_x)^k u \tag{1}$$

where k is a parameter. (Including phases for the variables requires only minor changes in the discussion to follow.) We are actually interested only in $0 \neq k \neq 1$, since $k = 0$ and $k = 1$ are easily understood and well known. So, upon changing the dependent variable to

$$v = \frac{k-1}{k} u^{k/(k-1t)} \tag{2}$$

equation (1) becomes

$$v_t = (v_x)^k. \tag{3}$$

Differentiating this with respect to x and making a further change of variable to

$$w = k(v_x)^{k-1} = k(u_x)^{k-1} u \tag{4}$$

we find the familiar Euler–Monge equation in the canonical form

$$w_t = w w_x. \tag{5}$$

Thus for any $k \neq 0$ and $k \neq 1$ the original equation (1) is reduced to Euler's through a change of dependent variable, although for technical reasons that are more or less obvious from the explicit constructions, it is often useful to assume $k > 1$. (For $k = 1$ of course, (1) is already the Euler–Monge equation *without* any change of variable.)

As is well-known (cf [1] or [2] for references) the general solution for w is given implicitly by

$$w = F(x + wt) \tag{6}$$

where F is an arbitrary differentiable function. By using the previous changes of variables and integrating once with respect to x, solutions for v and hence u, follow from those for w. For example, if F is linear, $w = \frac{x-x_0}{t_0-t}$, $v = \frac{k-1}{k}\frac{(x-x_0)^{k/(k-1)}}{[k(t_0-t)]^{1/(k-1)}}$ and $u = \frac{x-x_0}{[k(t_0-t)]^{1/k}}$.

Moreover, (1) leads to two infinite families of local conserved currents whose time and space components are powers of u and u_x, but not higher derivatives. The first family is quickly seen to be

$$\left(J_0^{(n)}, J_1^{(n)}\right) = (u^n u_x, u^{n+1}(u_x)^k) \tag{7}$$

for any n, not necessarily integer. Obviously, all these currents have simple topological charges. On the solution set of (1), $(n+1)J_\mu^{(n)} = \varepsilon_{\mu\nu}\partial^\nu u^{n+1}$, and $\partial^\mu J_\mu^{(n)} = 0$ immediately follows. The second family of currents may be obtained from the known (non-topological) conserved currents for w, namely $((n+1)w^n, nw^{n+1})$, just by changing variables. Thus[3]

$$\begin{aligned}\left(K_0^{(n)}, K_1^{(n)}\right) &= ((n+1)(v_x)^{n(k-1)}, nk(v_x)^{(n+1)(k-1)})\\ &= ((n+1)(u_x)^{n(k-1)}u^n, nk(u_x)^{(n+1)(k-1)}u^{n+1}).\end{aligned} \tag{8}$$

On the solution set of (1), or equivalently (3), it is straightforward to show $\partial^\mu K_\mu^{(n)} = 0$.

Finally, the linearization of (5) as given in [2] can be used to relate the spatial derivative of (3), or equivalently of (1), to a linear equation. Define

$$\psi(a, x, t) \equiv \frac{1}{a}(\exp(ak(v_x)^{k-1}) - 1). \tag{9}$$

Technically, it is useful to assume $k > 1$ here, especially for slowly varying v. It follows that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial a\partial x}\right)\psi = (v_t - (v_x)^k)_x \times k(k-1)(v_x)^{k-2}\exp(ak(v_x)^{k-1}). \tag{10}$$

Thus $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial a\partial x}\right)\psi = 0$ iff $(v_t - (v_x)^k)_x = 0$. (If the second factor on the RHS of (10) were to vanish, for both positive and negative a, this would require $k > 2$, $v_x = 0$, and hence also $(v_t - (v_x)^k)_x = 0$.) Encoding initial data for the nonlinear system in the form (9) therefore allows the data to be evolved linearly. Given a well-behaved solution to the linear equation for ψ, we may then extract the nonlinear data at other times just by taking the limit $v_x(x,t) = \left(\frac{1}{k}\lim_{a\to 0}\psi(a,x,t)\right)^{1/(k-1)}$. Integrating with respect to x modulo a function of only the time variable yields v, hence u.

Acknowledgments

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[3] In terms of u, as $k \to 1$ we note that $(n+1)(n+2)J_\mu^{(n)} \to \partial_x K_\mu^{(n+1)} \to ((n+2)(u^{n+1})_x, (n+1)(u^{n+2})_x)$.